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# A NOTE ABOUT TOPOLOGICALLY TRANSITIVE CYLINDRICAL CASCADES

BY

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ABSTRACT

Let $\sigma\colon \Sigma \to \Sigma$ be a topologically mixing shift of finite type. Let $\beta\colon \Sigma \to \mathbb{R}$ be a continuous function, and $\sigma_\beta$ be the skew-product of $\sigma$ by $\beta$. Assume that $\sigma_\beta$ has a positive semi-orbit that reaches any upper height, and any lower height. Then, arbitrarily $C^0$-close to $\beta$ there exists a Hölder map $\beta'\colon \Sigma \to \mathbb{R}$ such that the skew-product $\sigma_{\beta'}$ of $\sigma$ by $\beta'$ is topologically transitive.

## 1. Introduction

One of the important properties studied in the topological theory of dynamical systems is topological transitivity. Let $X$ be a topological space and $f\colon X \to X$ be a homeomorphism of $X$. Then $f$ is said to be **topologically transitive** (t.t.) if there exists a point $x_0 \in X$ with the trajectory $\mathcal{O}(x_0) = \{f^n(x_0) | n \in \mathbb{Z}\}$ everywhere dense. Such a point is called **topologically transitive**.

It was noticed for a long time that many dynamical systems with some hyperbolicity are topologically transitive. Our goal here is to contribute to the study of topologically transitive skew-products over shifts of finite type, when the fiber is the real line $\mathbb{R}$. We recall the definition of a skew-product.

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*Definition:* Let $X$ be a topological space, $G$ a topological group, $f\colon X \to X$ a homeomorphism, and $\beta\colon X \to G$ a continuous map. Then the map $f_\beta\colon X \times G \to X \times G$ given by $f_\beta(x,g) = (f(x), \beta(x)g)$, for $x \in X$, and $g \in G$, is called a **skew-product**. $X$ is called the **base**, and $G$ is called the **fiber** of the skew-product. The map $\beta$ is called a **cocycle**. If $G = \mathbb{R}$, we call the skew-product **cylindrical transformation**.

When the fiber is a compact connected Lie group, generic results about skew-products were obtained by Brin ([Br]). He showed that in the class of $C^2$ skew-products over a topologically transitive Anosov diffeomorphism, there is an open dense set of t.t. transformations. Recent related results about this class of transformations can be found in [BW], [D], [FP], [NT], [PP].

If the fiber is an unbounded manifold, topological transitivity is not very well understood. Nevertheless, there are known examples of t.t. transformations with unbounded fiber. Schnirelman and Besicovich ([Sh], [B]) found t.t. transformations of $\mathbb{R}^2 - \{0\}$ that, under a suitable change of coordinates, become a skew-product over an irrational rotation with fiber $\mathbb{R}$. Their construction is generalized in the book of Gottschalk and Hedlund ([GH]). Sidorov ([Si]) constructed examples of t.t. skew-products when the action in the base is any t.t. homeomorphism of a complete metric space, and the fiber is any separable Banach space. We note also the paper of Guivarc'h ([G]), where sufficient conditions for topological transitivity and (even ergodicity) are found. The results are proved using the local limit theorem of Guivarc'h and Hardy ([GuH]). This local limit theorem implies that if $(\Sigma, \sigma)$ is a shift of finite type, $m$ is a Gibbs measure, and $\beta\colon \Sigma \to \mathbb{R}^d$ has integral zero with respect to some Gibbs measure, and is not cohomologous to a cocycle with values in a proper coset, then for $A, B \subset \Sigma$ cylinders, and $a \in \mathbb{R}^d, \epsilon > 0$, the quantity

$$m\left(A \cup \sigma^{-n} B \cup \left\{x : \left|\sum_{k=0}^{n-1} \beta \circ \sigma^k(x) - a\right| < \epsilon\right\}\right)$$

is of order $1/n^{d/2}$ as $n \to \infty$. Hence the skew-product $\sigma_\beta$ is topologically mixing (and consequently topologically transitive). Guivarc'h notes also that for $n \geq 3$, the skew-product $\sigma_\beta$ cannot be ergodic because of dissipativity.

The following definition is needed.

*Definition:* Let $f_\beta\colon X \times \mathbb{R} \to X \times \mathbb{R}$ be a skew-product with fiber $\mathbb{R}$. Let $y \in X \times \mathbb{R}$. We say that the positive semiorbit $\mathcal{O}^+(y) = \{f^n(y) | n \in \mathbb{N}\}$ **reaches any upper (lower) height** if the projection $\mathrm{proj}_{\mathbb{R}}(\mathcal{O}^+(y)) \cap [0, \infty)$ (respectively $\mathrm{proj}_{\mathbb{R}}(\mathcal{O}^+(y)) \cap (-\infty, 0]$) is unbounded.

THEOREM 1.1: *Let $\sigma\colon \Sigma \to \Sigma$ be a topologically mixing shift of finite type. Let $\beta\colon \Sigma \to \mathbb{R}$ be a continuous function, and let $\sigma_\beta\colon \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}$ be the skew-product given by $\sigma_\beta(\omega, t) = (\sigma\omega, \beta(\omega) + t)$. Assume that $\sigma_\beta$ has a positive semi-orbit that reaches any upper height, and any lower height. Then, arbitrarily $C^0$-close to $\beta$ there is a Hölder map $\beta'\colon \Sigma \to \mathbb{R}$ such that the skew-product $\sigma_{\beta'}$ is topologically transitive.*

In general, the existence of a positive semi-orbit that reaches any upper height, and any lower height, does not imply that the skew-product is topologically transitive. We will show such an example before the proof of Theorem 1.1. Note also that the weaker assumption that $\beta$ has an *orbit* that reaches any upper height, as well as any lower height, is not sufficient to reach the conclusion of Theorem 1.1. Indeed, a counterexample can be easily obtained by considering the skew-product given by a strictly positive cocycle.

This paper emerged from an attempt to find sufficient conditions for the existence of topologically transitive skew-products over Anosov diffeomorphisms, when the fiber is $\mathbb{R}$. We do not know if the result in Theorem 1.1 is true if the map in the base is an Anosov diffeomorphism, instead of a shift. Note that if $X$ is a compact metrizable space that is perfect, then a homeomorphism $f\colon X \to X$ is t.t. if and only if there is a point $y \in X$ whose positive semi-orbit $\mathcal{O}^+(y)$ is dense (see Exercise 1.4.2 in [KH]). This property is false if $X$ is a non-compact topological space.

The following conjecture seems natural.

CONJECTURE: *Let $M$ be a compact manifold and $A\colon M \to M$ a transitive $C^1$ Anosov diffeomorphism. Let $\beta\colon M \to \mathbb{R}$ be a Hölder function, and let $A_\beta\colon M \times \mathbb{R} \to M \times \mathbb{R}$ be the skew-product given by $A_\beta(x, y) = (Ax, \beta(x) + y)$. Assume that $A_\beta$ has a positive semi-orbit that reaches any upper height, as well as any lower height. Then $A_\beta$ is topologically transitive.*

## 2. Proof of the main result

Let $k \geq 2$, and let $A$ be a $k \times k$ $0-1$ matrix. Define

$$\Sigma = \Sigma_A = \left\{ \omega = (\omega_n)_{-\infty}^{\infty} \in \prod_{n=-\infty}^{\infty} \{1, \dots, k\} | A(\omega_n, \omega_{n+1}) = 1 \text{ for all } n \in \mathbb{Z} \right\}.$$

The elements of the set $\{1, 2, \dots, k\}$ are called **symbols**.

The map $\sigma\colon \Sigma \to \Sigma$ given by

$$(\sigma\omega)_n = \omega_{n+1}$$

is called a **shift of finite type**. This is a homeomorphism of $\Sigma$ with respect to the Tychonov product topology. Note that $\sigma$ is **topologically mixing** (i.e., when $U, V$ are open sets in $\Sigma$, there is $N$ such that $\sigma^m U \cap V \neq \emptyset$ for any $m \geq N$) if and only if there is $M > 0$ such that $A^M(i,j) > 0$ for any $1 \leq i, j \leq k$.

Fix an integer $N \geq 0$ and symbols $\alpha_{-N}, \dots, \alpha_N$ and call the subset

$$C^{-N,\dots,N}_{\alpha_{-N},\dots,\alpha_N} = \{\omega \in \Sigma | \omega_{n_i} = \alpha_i \text{ for } i = -N, \dots, N\}$$

a **(symmetric) cylinder of rank N**. For any positive integer $N$ we define a partition of $\Sigma$ given by the set of symmetric cylinders $C^{-N,\dots,N}_{\alpha_{-N},\dots,\alpha_N}$ of rank $N$. For future reference, this partition is called **the $N$-th Markov partition of $\sigma$**. To any shift and any fixed Markov partition $\mathcal{P}$, we associate an oriented graph denoted $\mathcal{G}$. We identify cylinders in the Markov partition with the vertices of the graph, and connect two vertices by an oriented arrow if there are points in the first cylinder that are mapped into the second cylinder by the shift $\sigma$. A finite sequence of vertices of the graph is called an **admissible path** if any two consecutive vertices in the sequence are connected by an oriented arrow. A closed admissible path is called a **cycle**. A cycle is called **minimal** if any vertex different from the end-points of the path appears only once in the path.

For any $\lambda > 1$, $\Sigma$ becomes a metric space with the metric given by

$$d_\lambda(\omega^1, \omega^2) = \sum_{n=-\infty}^{\infty} \frac{|\omega^1_n - \omega^2_n|}{\lambda^{|n|}}.$$

The metrics $d_\lambda$ define the same topology on $\Sigma$. We fix $\lambda$ for the rest of the paper.

For $0 < \theta < 1$ denote by $C_\theta(\Sigma, \mathbb{R})$ the space of $\theta$-Hölder functions from $\Sigma$ to $\mathbb{R}$. For $h \in C_\theta(\Sigma, \mathbb{R})$, there is a constant $C > 0$ such that

$$|h(\omega^1) - h(\omega^2)| \leq C d_\lambda(\omega^1, \omega^2)^\theta,$$

for $\omega^1, \omega^2 \in \Sigma$.

Let $h$ be a continuous real valued function defined on $\Sigma$. For $n = 0, 1, \dots$ let

$$V_n(h) := \max\{|h(\omega) - h(\omega')| | \omega_k = \omega'_k \text{ for } |k| \leq n\}.$$

The function $h$ is said to be of exponential type if there are constants $a, c > 0$ such that

$$V_n(h) \leq c 2^{-an}.$$

It is not difficult to see that $h$ is of exponential type if and only if it is Hölder continuous with respect to the metric $d_\lambda$ for some $\lambda$.

Let $M = \Sigma \times \mathbb{R}$. $M$ is a complete metric space with the product metric denoted $d_M$.

For $\omega \in \Sigma$ we will consider the **stable** and *unstable leaves* of $\sigma$:

$$W^s(\omega) = \{\bar{\omega} | \text{ there is } n_0 \text{ such that } \bar{\omega}_i = \omega_i, \text{ for } i \geq n_0\},$$
$$W^u(\omega) = \{\bar{\omega} | \text{ there is } n_0 \text{ such that } \bar{\omega}_i = \omega_i, \text{ for } i \leq n_0\}.$$

Note that $\{W^s(\omega) | \omega \in \Sigma\}$ and $\{W^u(\omega) | \omega \in \Sigma\}$ are partitions of $\Sigma$. These partitions are $\sigma$-invariant. If $\bar{\omega} \in W^{s(u)}(\omega)$, then $d_\lambda(\sigma^n \omega, \sigma^n \bar{\omega}) \to 0$ as $n \to \infty(-\infty)$, exponentially fast.

Let $\beta \in C_\theta(\Sigma, \mathbb{R})$, $\omega \in \Sigma$, and $\mathcal{O} = \{\omega, \sigma(\omega), \ldots, \sigma^n(\omega)\}$ be a *trajectory*. Then

$$\beta(n+1, \omega) := \beta(\omega) + \beta(\sigma(\omega)) + \cdots + \beta(\sigma^n(\omega)) \tag{2.1}$$

is called the **height** of $\sigma_\beta$ over $\mathcal{O}$.

For $\bar{\omega} \in W^s(\omega)$ define

$$\gamma_s(\omega, \bar{\omega}) = \lim_{N \to \infty} [\beta(N, \omega) - \beta(N, \bar{\omega})],$$

and for $\bar{\omega} \in W^u(\omega)$ define

$$\gamma_u(\omega, \bar{\omega}) = \lim_{N \to \infty} [\beta(N, \sigma^{-N}\omega) - \beta(N, \sigma^{-N}\bar{\omega})].$$

Both limits are convergent because $\beta$ is Hölder.

Consider the skew-product $\sigma_\beta$. For $(\omega, t) \in M$ we will consider the **stable** and **unstable leaves** of $\sigma_\beta$:

$$\widetilde{W}^s(\omega, t) = \{(\bar{\omega}, \tau) | \bar{\omega} \in W^s(\omega), \tau = t + \gamma_s(\omega, \bar{\omega})\},$$
$$\widetilde{W}^u(\omega, t) = \{(\bar{\omega}, \tau) | \bar{\omega} \in W^u(\omega), \tau = t + \gamma_u(\omega, \bar{\omega})\}.$$

Note that $\{\widetilde{W}^s(\omega, t) | (\omega, t) \in \Sigma \times \mathbb{R}\}$ and $\{\widetilde{W}^u(\omega, t) | (\omega, t) \in \Sigma \times \mathbb{R}\}$ are partitions of $\Sigma \times \mathbb{R}$.

It is easy to see that for $(\bar{\omega}, \tau) \in \widetilde{W}^{s(u)}(\omega, t)$, one has $d_M(\sigma_\beta^n(\omega, t), \sigma_\beta^n(\bar{\omega}, \tau)) \to 0$ as $n \to \infty(-\infty)$, exponentially fast. Note that the partitions of $M$ given by $\widetilde{W}^s$ and $\widetilde{W}^u$ are $\sigma_\beta$-invariant.

A **chain** in $\Sigma$ is a set of points $\{\omega^0, \omega^1, \ldots, \omega^n\}$ such that for all $i$ either $\omega^{i+1} \in W^s(\omega^i)$, or $\omega^{i+1} \in W^u(\omega^i)$. If $\mathcal{C}_1 = \{\omega^0, \omega^1, \ldots, \omega^k\}$ and $\mathcal{C}_2 = \{\pi^0, \pi^1, \ldots, \pi^l\}$ are two chains such that $\omega^k = \pi^0$, $\mathcal{C}_1$ and $\mathcal{C}_2$ can be juxtaposed in the chain

$$\mathcal{C}_1\mathcal{C}_2 = \{\omega^0, \omega^1, \ldots, \omega^{k-1}, \pi^1, \ldots, \pi^l\}.$$

To any chain $\mathcal{C} = \{\omega^0, \omega^1, \ldots, \omega^k\}$, one can associate the **opposite chain**

$$-\mathcal{C} = \{\omega^k, \omega^{k-1}, \ldots, \omega^0\}.$$

One can also define chains in $M$. We say that the chain

$$\widetilde{\mathcal{C}}(t_0) = \{(\omega^0, t_0), (\omega^1, t_1), \ldots, (\omega^n, t_n)\}$$

covers the chain $\mathcal{C} = \{\omega^0, \omega^1, \ldots, \omega^n\}$, and call it the **lifted chain of $\mathcal{C}$ starting at $t_0$**. For any such chain there exists a number $h(\widetilde{\mathcal{C}}) \in \mathbb{R}$, called the **height** of $\widetilde{\mathcal{C}}$, such that $t_n = h(\widetilde{\mathcal{C}}) + t_0$. If the chain $\mathcal{C}$ is fixed, $h(\widetilde{\mathcal{C}})$ is independent of $t_0$. Note that $h(-\widetilde{\mathcal{C}}) = -h(\widetilde{\mathcal{C}})$.

The height over a chain of type $Q = \{\omega^0, \omega^1, \omega^0\}$ that satisfies $\omega^1 \in W^s(\omega^0)$, $\omega^1 \in W^u(\omega^0)$ is given by the formula

$$(2.2) \quad h(\widetilde{Q}) = \lim_{N\to\infty} [(\beta(N, \omega^0) - \beta(N, \omega^1)) + (\beta(N, \sigma^{-N}\omega^1) - \beta(N, \sigma^{-N}\omega^0))].$$

Assume now that $(X, d_X)$ is a complete metric space, and $f\colon X \to X$ a homeomorphism. We recall below a few facts from topological dynamics.

Let $W$ be a partition of $X$. Denote by $W(x)$ the element of the partition containing $x \in X$. $W$ is called **$f$-invariant** if $f(W(x)) \subset W(f(x))$, for any $x \in X$. An $f$-invariant partition $W$ is called **contracting** if for any $x \in X$ and $y \in W(x)$ we have $\lim_{N\to\infty} d_X(f^N(x), f^N(y)) = 0$. An $f$-invariant partition is called **expanding** if it is contracting for $f^{-1}$.

Fix now $W_1, W_2$ two partitions of $X$. A subset $S \subset X$ is called **accessible** if for any pair of points $x, y \in S$ there is a positive integer $N$, and a chain $\{x_1, x_2, \ldots, x_N\} \subset X$, such that $x_1 = x, x_N = y, x_{i+1} \in W_j(x_i), i = 1, 2, \ldots, N-1, j = 1, 2$. If $\epsilon > 0$, the subset $S$ is called **$\epsilon$-accessible** if instead of $x_N = y$ we have $d(x_N, y) < \epsilon$.

*Remark:* Note that a set $S$ can be $\varepsilon$-accessible for any $\varepsilon > 0$, without being accessible. An example is given by an ergodic toral automorphism $f\colon \mathbb{T}^n \to \mathbb{T}^n$ that is partially hyperbolic but not hyperbolic. The automorphism $f$ has a pair of invariant smooth foliations, called stable and unstable, that have each leaf dense. So, for $\varepsilon > 0$, $\mathbb{T}^n$ is $\varepsilon$-accessible with respect to each foliation. Nevertheless, the direct sum of the invariant distributions tangent to the foliations is of dimension strictly less than $n$, and it is integrable. Any accessible set is contained in a leaf of the integral foliation, so it is of measure 0.

A partition $W$ is called **of foliation type** if for any $x \in X$, $y \in W(x)$, and any $\epsilon > 0$, there is $\bar{\epsilon} = \bar{\epsilon}(x, y, \epsilon)$ such that if $C$ is a non-empty open set included in the ball $B(x,\bar{\epsilon})$, then the set $B(y,\epsilon)\cap\left(\bigcup_{z\in C}W^s(z)\right)$ contains a non-empty open set.

The following criteria for topological transitivity is well known (see [KH], Lemma 1.4.2).

LEMMA 2.1: *Let $f\colon X\to X$ be a continuous map of a locally compact separable space. The map $f$ is t.t. if and only if for any two nonempty open sets $U,V\subset X$ there is an integer $m=m(U,V)$ such that $f^m(U)\cap V$ is nonempty.*

We need also the following result of Brin (see [Br]). Because our statement is more general, and for the reader's convenience, we give the proof below.

THEOREM 2.2: *Let $(X,d_X)$ be a complete metric space, $f\colon X\to X$ a homeomorphism of $X$, and $(W^s,W^u)$ a pair of $f$-invariant partitions of $X$, both of foliation type, $W^s$ contracting and $W^u$ expanding. Assume that:*

1. *$X$ is $\epsilon$-accessible for any $\epsilon>0$;*
2. *the map $f$ is topologically recurrent, i.e., for any open ball $B\subset X$, there exists an integer $n\neq 0$ such that $B\cap f^nB\neq 0$.*

*Then the map $f$ is topologically transitive.*

*Proof:* We use Lemma 2.1. Let $U,V\subset X$ be non-empty open sets. Fix $x,y\in X$ and $\delta>\epsilon>0$ such that $B(x,\delta)\subset V, B(y,\delta)\subset U$. Because $X$ is $\epsilon$-accessible, for any pair of points $x,y\in X$, there exist points $x_1,x_2,\dots,x_n\in X$ such that $\mathrm{dist}_X(x_1,y)<\epsilon$, $x_1=x, x_{i+1}\in W^j(x_i)$, for $i=1,2,\dots,n-1$, $j=s$ or $u$, and $x_N=x$. It is enough to prove that there exists an integer $m$ such that $f^mB(y,\delta)\cap B(x,\delta)$ is not empty. The following two lemmas are necessary for this.

LEMMA 2.3: *Let $x_{i+1}\in W^s(x_i)$. Then for any $\varepsilon_1>0$ there exists $\epsilon_2>0$ such that if the set $C:=f^{k_0}B(x,\delta)\cap B(x_i,\varepsilon_2)$ is nonempty for some integer $k_0$, then one can find an integer $k$ such that the set $f^kB(x,\delta)\cap B(x_{i+1},\varepsilon_1)$ is nonempty.*

*Proof:* The partition $W^s$ is of foliation type, so there is $\epsilon_2:=\bar{\epsilon}(x_i,x_{i+1},\epsilon_1/2)$ such that the set

$$C_1:=B\left(x_{i+1},\frac{\varepsilon_1}{2}\right)\cap\Big(\bigcup_{z_1\in C}W^s(z_1)\Big)$$

contains a non-empty open set $C_2$. From the assumption (2) in the Theorem 2.2 there exists a point $z\in C_2$ and integers $m$ arbitrarily large such that $f^m(z)\in C_2$.

Let $D=W^s(z)\cap f^{k_0}B(x,\delta)$. It is clear that the set $D$ is nonempty. Indeed, by the definition of $C_2$, there is $z_1\in f^{k_0}B(x,\delta)$ such that $z\in W^s(z_1)$, and

$W^s(z_1) = W^s(z)$. Since $W^s$ is a contracting partition, there exists $m_0 > 0$ such that for any $m \geq m_0$, the set $f^m D \cap B(f^m z, \varepsilon_1/4)$ is nonempty. We choose now $m \geq m_0$ such that $d_X(f^m z, z) < \varepsilon_1/4$. We have that

$$d_X(z, f^m D) < d_X(z, f^m z) + d_X(f^m z, f^m D) < \frac{\varepsilon_1}{2}.$$

Since $d_X(z, x_{i+1}) < \varepsilon_1/2$, it follows that $d_X(f^m D, x_{i+1}) < \varepsilon_1$. Thus the set $f^m D \cap B(x_{i+1}, \varepsilon_1)$ is nonempty, and therefore the set $f^{m+k_0} B(x,\delta) \cap B(x_{i+1}, \varepsilon_1)$ is nonempty (because $D \subset f^{k_0} B(x,\delta)$). ∎

LEMMA 2.4: *Let $x_{i+1} \in W^u(x_i)$. Then for any $\varepsilon_1 > 0$, there exists $\varepsilon_2 > 0$ such that if the set $C := f^{k_0} B(x,\delta) \cap B(x_i, \varepsilon_2)$ is nonempty for some integer $k_0$, then there is an integer $k$ such that the set $f^k B(x,\delta) \cap B(x_{i+1}, \varepsilon_1)$ is nonempty.*

*Proof:* The proof is similar to the proof of Lemma 2.3, but now one has to consider backward iterates of $f$. ∎

*The end of the proof of Theorem 2.2:* Since $\epsilon < \delta$, the ball $B(y,\delta)$ contains $x_1$. Apply now alternatively Lemma 2.3 and Lemma 2.4 to show that there is an integer $m$ such that $f^m B(y,\delta) \cap B(x,\delta)$ is non-empty. ∎

The perturbations we consider in order to prove Theorem 1.1 are functions constant on the cylinders of a given $N$-th Markov partition $\mathcal{P}$. We will call such cocycles **locally constant**. A locally constant cocycle is Hölder.

If the cocycle $\beta$ is locally constant, and if the endpoints of the trajectory $\mathcal{O}$ coincide, then the height of $\beta$ along $\mathcal{O}$ depends only on the cycle determined by $\mathcal{O}$ in the graph $\mathcal{G}$.

A finite sequence of symbols $B$ is called a **block**. The number of symbols in the block $B$ is called the **length** of the block, and is denoted by $l(B)$. If for any two consecutive elements $\omega_i, \omega_{i+1}$ in the block we have $A(\omega_i, \omega_{i+1}) = 1$, the block is called **admissible**. If we fix an entry $E$ in the block, the block is called **centered**. If a centered block $(B, E)$ consists of consecutive entries in an element $\omega \in \Sigma$, then the entry $E$ corresponds to $\omega_0$. Given an arbitrary pair of blocks $B_1, B_2$, we can juxtapose them and obtain the block $B_1 B_2$. If $B_1$ is centered, then $B_1 B_2$ is centered and the fixed entry in $B_1 B_2$ is the fixed entry in $B_1$. If $S$, $b_1$ and $b_2$ are blocks such that $B_1 = b_1 S$ and $B_2 = S b_2$, then we can also define $B_1 \circ_S B_2$ to be the block $b_1 S b_2$. Note that if $\sigma$ is topologically mixing, and $N$ is an integer big enough, any two symbols can be joined by a block of length $N$.

If $\beta$ is a locally constant cocycle, and if $B$ is a block of length bigger than $2N+1$, then we will define the **height of $\beta$ over the block $B$**. Indeed, the first

$2N+1$ symbols in the block determine a cylinder. By shifting a frame of length $2N+1$ along the block, we obtain $l(B)-2N$ cylinders in the Markov partition. The height of $\beta$ over the block $B$, denoted by $h_\beta(B)$, is the sum of the values of $\beta$ over the first $l(B)-2N-1$ of these cylinders. Note also that any cycle of length $k$ in the graph $\mathcal{G}$ determines a block of length $2N+1+k$. Indeed, any vertex in the cycle determines a cylinder, and so a block of dimension $2N+1$. We can build the block of length $2N+1+k$ by overlapping any two consecutive blocks given by the cycle over a block of dimension $2N$. The correspondence between blocks and cycles allow us to define the height of $\beta$ over a cycle $K$ in $\mathcal{G}(\mathcal{P})$. Note that in the computation of the height over $K$, the value of $\beta$ over the last cylinder in $K$ is not considered. In what follows, we will identify sometimes a cylinder with the block that determines the cylinder, or with a vertex in $\mathcal{G}(\mathcal{P})$.

The stable and unstable partitions defined above for the skew-product $\sigma_\beta$ are of foliation type. We show the proof when $\beta$ is locally constant.

LEMMA 2.5: *Let $\sigma\colon \Sigma \to \Sigma$ be a shift of finite type, and $\beta\colon \Sigma \to \mathbb{R}$ a locally constant cocycle. Then the stable and unstable partitions $\widetilde{W}^s$ and $\widetilde{W}^u$ are of foliation type.*

*Proof:* It is enough to show that $\widetilde{W}^s$ is of foliation type.

Let $x=(\omega^2,t_2)\in\Sigma\times\mathbb{R}, y=(\omega^1,t_1)\in\widetilde{W}^s(x)$, and $\varepsilon>0$. Without loss of generality, we can assume that $B(y,\varepsilon)=C_1\times(a_1,a_2)$, where $C_1$ is a cylinder and $(a_1,a_2)$ is an interval in $\mathbb{R}$. By taking $\bar\varepsilon$ small enough, we can assume that the open set $C\subset B(x,\bar\varepsilon)$ contains an open set of type $C_2\times(b_1,b_2)$, where $C_2$ is a cylinder of length much bigger than $C_1$. Choose two blocks $B_1,B_2$ such that $l(B_1C_1B_2)=l(C_2)$, and define the cylinder $C_3$ to be the $B_1C_1B_2$. Define a map $\phi$ from the cylinder $C_2$ into the cylinder $C_3$ that substitutes the block $C_2$ by the block $C_3$ in any $\omega\in C_2$. Note that $\gamma_s(\omega,\phi(\omega))$ is constant for $\omega\in C_2$. Denote the constant by $\gamma$. Since the point $(\phi(\omega),t+\gamma)$ is in the stable foliation of $(\omega,t)$, it follows that the open set $C_3\times(b_1,b_2)$ is in the intersection $B(y,\epsilon)\cap\left(\bigcup_{z\in C}W^s(z)\right)$, if $C=C_2\times(b_1-\gamma,b_2-\gamma)$. ∎

Note also that the partition $\widetilde{W}^s$ is contracting, $\widetilde{W}^u$ is expanding, and both partitions are $\sigma_\beta$ invariant. Therefore Theorem 1.1 is proved if we find an arbitrarily $C^0$-small perturbation that satifies conditions (1), (2) and (3) in Theorem 2.2.

*Definition:* We call a pair $(a,b)$ of real numbers **independent** if the set $\mathbb{N}a+\mathbb{N}b$ is dense in $\mathbb{R}$.

The next lemma is needed.

LEMMA 2.6: *Let $a, b \in \mathbb{R}$, $a > 0$, $b < 0$. Assume that $a/b$ is irrational. Then the pair $(a, b)$ is independent.*

*Proof:* Use first the fact that an irrational rotation of the circle is minimal to show that $-a$ and $-b$ are in the closure of $\mathbb{N}a + \mathbb{N}b$. Then apply the fact again to show that $\mathbb{N}a + \mathbb{N}b$ is dense in $\mathbb{R}$. ∎

The following lemmas show sufficient conditions for a skew-product by a locally constant cocycle to satisfy the conditions in Theorem 2.2.

LEMMA 2.7: *Let $\sigma: \Sigma \to \Sigma$ be a topologically mixing shift of finite type, and $\mathcal{P}$ a Markov partition of $\sigma$. Let $\beta: \Sigma \to \mathbb{R}$ be a locally constant cocycle. Assume that there are two cycles $K_1, K_2$ in the graph $\mathcal{G}(\mathcal{P})$, starting at the same vertex $C_0$, such that the height $h_1$ of $\beta$ over one of them is positive, the height $h_2$ of $\beta$ over the other is negative, and $h_1/h_2$ is irrational.*

*Then the skew-product $\sigma_\beta$ satisfies condition (2) in Theorem 2.2.*

*Proof:* Let $\mathcal{P}$ be the $N$-th Markov partition, let $\epsilon > 0$ and $x_0 = (\omega^0, t_0) \in \Sigma \times \mathbb{R}$. Consider the ball

$$B(x_0, \epsilon) = \{(\omega', t') \in \Sigma \times \mathbb{R} | d_\lambda(\omega^0, \omega') < \epsilon, |t_0 - t'| < \epsilon\}.$$

We have to show that there is an integer $n$ and $(\omega, t) \in B(x_0, \epsilon)$ such that $\sigma_\beta^n(\omega, t) \in B(x_0, \epsilon)$. For $\epsilon > 0$, there is a positive integer $n_0$ such that $\omega_i = \omega_i^0$ for $|i| \leq n_0$ implies that $d_\lambda(\omega, \omega^0) < \epsilon$. We can assume that $n_0 > N$.

Define the blocks

$$B_0 = (\omega_{-n_0}^0, \ldots, \omega_0^0, \omega_1^0, \ldots, \omega_{n_0}^0),$$
$$B_0' = (\omega_{-N}^0, \ldots, \omega_0^0, \omega_1^0, \ldots, \omega_{n_0}^0),$$
$$B_0'' = (\omega_{-n_0}^0, \ldots, \omega_0^0, \omega_1^0, \ldots, \omega_N^0).$$

Assume that the block $B_0$ is centered at $\omega_0^0$. Denote by $B_i$ the block determined by the cycle $K_i$, $i = 1, 2$. Denote by $j_0$ the first symbol of $B_1$, and by $k_0$ the last symbol of $B_2$. Because the shift $\sigma$ is topologically mixing, we can find two blocks $B_3$ and $B_4$ such that the blocks $\omega_{n_0}^0 B_3 j_0$ and $k_0 B_4 \omega_{-n_0}^0$ are admissible. Denote by $B_5$ the block determined by the cylinder $C_0$. If $n_1, n_2$ are positive integers, define the blocks $(n_1 B_1)$ and $(n_2 B_2)$ to be

$$(n_1 B_1) := B_1 \circ_{B_5} B_1 \circ_{B_5} B_1 \circ_{B_5} \ldots \circ_{B_5} B_1,$$
$$(n_2 B_2) := B_2 \circ_{B_5} B_2 \circ_{B_5} B_2 \circ_{B_5} \ldots \circ_{B_5} B_2.$$

The block $B_1$ appears $n_1$ times, and the block $B_2$ appears $n_2$ times. Define now the block

$$B(n_1,n_2) := B_0' B_3(n_1B_1) \circ_{B_5} (n_2B_2)B_4B_0''.$$

Then the height of $\beta$ over $B(n_1,n_2)$ is given by

$$h_\beta(B(n_1,n_2)) := n_1h_1 + n_2h_2 + \nu,$$

where $\nu$ is a constant independent of $n_1, n_2$.

Using Lemma 2.6, it follows that there are arbitrarily large positive integers $n_1$ and $n_2$ such that

$$|h_\beta(B(n_1,n_2)) - t_0| < \epsilon.$$

For $n_1, n_2$ large enough, define $\omega$ to be any element in $\Sigma$ containing the centered block

$$B_0B_3(n_1B_1) \circ_{B_5} (n_2B_2)B_4B_0,$$

and define $t = t_0$. ∎

Lemma 2.8: *Let $\sigma\colon \Sigma \to \Sigma$ be a topologically mixing shift of finite type, and $\mathcal{P}$ a Markov partition of $\sigma$. Let $\beta\colon \Sigma \to \mathbb{R}$ be a locally constant cocycle. Assume that there are two chains $Q_1, Q_2$ in $\Sigma$ that have a common vertex $\omega^0$. Assume also that the pair of heights $(h(\widetilde{Q}_1), h(\widetilde{Q}_2))$ is independent. Then the skew-product $\sigma_\beta$ satisfies condition (1) in Theorem 2.2.*

*Proof:* It follows from the fact that $\sigma$ is topologically mixing that the set $\Sigma$ is accessible with respect to the pair of partitions $(W^s, W^u)$. From this it follows that in order to prove the lemma it is enough to show that a fiber is $\epsilon$-accessible.

We will show that the fiber over the common vertex $\omega^0$ is $\epsilon$-accessible for any fixed $\epsilon > 0$. Fix two points in the fiber, say $(\omega^0, t_1), (\omega^0, t_2)$. It follows from Lemma 2.6 that there are integers $n_1, n_2$ such that

$$|n_1h(\widetilde{Q}_1) + n_2h(\widetilde{Q}_2) - (t_2 - t_1)| < \epsilon. \tag{2.2}$$

Consider now in the base the chain $P = (n_1 \text{ times } Q_1)(n_2 \text{ times } Q_2)$. Then the height of the lifted chain $\widetilde{P}(\omega^0, t_1)$ is $n_1h(\widetilde{Q}_1) + n_2h(\widetilde{Q}_2)$. So it follows from Lemma 2.5 that by choosing $n_1, n_2$ appropriately, $P$ becomes a chain that makes the pair $(\omega^0, t_1), (\omega^0, t_2)$ $\epsilon$-accessible. ∎

*Remark:* We show now an example of a skew-product that has a positive semi-orbit that reaches any upper height, and any lower height. Our example is not topologically transitive. According to Theorem 1.1 one can find arbitrarily $C^0$-close to it topologically transitive skew-products. We will show the perturbation below. Indeed, consider $(\Sigma_A, \sigma_A)$ to be the shift of finite type defined by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. The 0-th Markov partition has two cylinders, $C_0$ and $C_1$. Define the cocycle $\beta\colon \Sigma \to \mathbb{R}$ to be $\beta(\omega) = 1$ if $\omega \in C_0$, and $\beta(\omega) = -1$ if $\omega \in C_1$. Consider two real numbers $a, b$ such that $a/b$ is irrational and $ab < 0$. Define a new cocycle $\beta'\colon \Sigma \to \mathbb{R}$ to be $\beta'(\omega) = a$ if $\omega \in C_0$, and $\beta'(\omega) = b$ if $\omega \in C_1$. Lemmas 2.7 and 2.8 can be used to show that the skew-product $\sigma_{\beta'}$ satisfies the conditions of Theorem 2.2, and thus is topologically transitive. Since $a$ can be choosen arbitrarily close to 1, and $b$ can be choosen arbitrarily close to $-1$, it follows that $\beta'$ can be arbitrarily $C^0$-close to $\beta$.

*Proof of Theorem 1.1:* Let $\epsilon > 0$. Choose an $N$-th Markov partition $\mathcal{P}$ such that the variation of $\beta$ over any cylinder in $\mathcal{P}$ is smaller than $\epsilon/4$. We will assume also that $N$ is bigger than $10\max\{l(B_1), l(B_2), l(B_3)\}$, where $B_i$ are blocks that are defined below.

Let $x_0 = (\omega^0, t_0)$ be the point in $\Sigma \times \mathbb{R}$ with the positive semi-orbit that reaches any upper height, as well as any lower height. For any cylinder $C_i \in \mathcal{P}$, choose a point $\omega_i \in C_i$ and define $\beta'(\omega) = \beta(\omega_i)$, if $\omega \in C_i$. Denote by $\mathcal{G}$ the subgraph determined in $\mathcal{G}(\mathcal{P})$ by $\omega^0$. We can assume, without loss of generality, that any vertex in $\mathcal{G}$ is visited infinitely many times by the orbit $\mathcal{O}(\omega_0)$. We claim that by making a perturbation of $\beta'$ within distance $\epsilon/4$, we can ensure that there exists a vertex $C_0$ in $\mathcal{G}$, and two cycles $K_1$ and $K_2$ in $\mathcal{G}$, that have $C_0$ common vertex, such that $h_{\beta'}(K_1)$ is strictly positive, and $h_{\beta'}(K_2)$ is strictly negative.

Assume that for any vertex $C_i$ in $\mathcal{G}$, the heights of $\beta'$ over all the cycles starting at $C_i$ are either all non-positive, or all non-negative. Because the graph $\mathcal{G}$ is connected, for any pair of vertices in $\mathcal{G}$ there is a cycle containing both vertices. We can assume, without loss of generality, that for any vertex $C_i$ in $\mathcal{G}$, the height of $\beta'$ over all the cycles starting at $C_i$ are non-negative. Consider now the heights of $\beta$ over all the minimal cycles in $\mathcal{G}$. Observe that, because the positive semi-orbit $\{\sigma_\beta^n(x_0) | n \in \mathbb{N}\}$ reaches any lower level, there exists a minimal cycle $K$ such that $h_\beta(K)$ is strictly negative. We can change the values of $\beta'$ within $\epsilon/4$, over the cylinders that are vertices of $K$, such that the height $h_{\beta'}(K)$ is strictly negative. After this perturbation, we look again at the heights of $\beta'$ over all the cycles in $\mathcal{G}$. If all the heights are non-positive, we define, at $C^0$-distance $\epsilon/2$ from $\beta$, a continuous family of cocycles $\beta'_t$, $0 \leq t \leq 1$, such that $\beta'_0$ has all the heights non-positive, and $\beta_1'$ has all the heights non-negative. It follows that, for a certain value of the parameter $t$, either there are two cycles $K_1, K_2$ starting at the same vertex $C_0$, such that $h_{\beta'}(K_1) > 0$, $h_{\beta'}(K_2) < 0$, or all the heights are zero. Assume that all the heights are zero. Note that $\mathcal{G}$ itself cannot be a minimal cycle, because otherwise $\omega^0$ is periodic, and the positive-semiorbit of $x_0$ under $\beta$ is either bounded from below or bounded from above, in contradiction with our assumption about $x_0$. It follows that $\mathcal{G}$ has at least two minimal cycles, $K_1$ and $K_2$, starting at the same vertex, and at least two vertices $C_1, C_2$, such that $C_i$ is a vertex of $K_i$, and $C_1$ is not a vertex of $K_2$. Because all the heights are zero, we can change the value of $\beta'$ within $\epsilon/8$ over the cylinder $C_1$ such that the height over $K_1$ becomes stricly positive, and then change the value of $\beta'$ within $\epsilon/16$ over the cylinder $C_2$, keeping the height over $K_1$ strictly positive, and making the height over $K_2$ stricly negative. Note also that we can assume that after the perturbation, the ratio $h_{\beta'}(K_1)/h_{\beta'}(K_2)$ is irrational.

Therefore Lemma 2.8 can be applied to show that $\beta'$ satisfies condition (1) in Theorem 2.2. For future reference, denote the height of $\beta'$ over $K_1$ by $h_1$, and the height of $\beta'$ over $K_2$ by $h_2$.

In what follows we show that one can make a new arbitrarily small perturbation of $\beta'$, keeping the conditions in Lemma 2.8 satisfied, such that the assumptions of Lemma 2.7 are also satisfied. This is achieved in two steps. We first make the heights $h(\widetilde{Q}_1)$ and $h(\widetilde{Q}_2)$ different from zero. Then, by an additional arbitrary small perturbation, we make the ratio $h(\widetilde{Q}_1)/h(\widetilde{Q}_2)$ irrational.

Consider the 0-th-Markov partition of $\sigma$, and the corresponding oriented graph. The vertices of the graph are symbols. Because $\sigma$ is topologically mixing, we can find two distinct minimal cycles $L_1$ and $L_2$, starting at the same vertex, say $s$. There exist blocks $b_1, b_2$, at least one of which is nontrivial, such that $L_1 = sb_1s$ and $L_2 = sb_2s$. Note that the symbols that appear in $b_1, b_2$ are all distinct, and different from $s$.

Consider the the blocks

$$B_1 = sb_1s, \quad B_2 = sb_2s, \quad B_3 = sb_1sb_2s.$$

Consider the following elements in $\Sigma$, all centered at $\underline{s} = s$:

$$\begin{aligned} \omega_0 &= \dots B_1 \circ_s B_1 \circ_s B_1 \circ_{\underline{s}} B_1 \circ_s B_1 \circ_s B_1 \dots, \\ \omega_1 &= \dots B_1 \circ_s B_1 \circ_s B_1 \circ_{\underline{s}} B_2 \circ_s B_1 \circ_s B_1 \dots, \\ \omega_2 &= \dots B_1 \circ_s B_1 \circ_s B_1 \circ_{\underline{s}} B_3 \circ_s B_1 \circ_s B_1 \dots, \end{aligned} \tag{2.3}$$

and the chains $Q_1 = \{\omega_0, \omega_1, \omega_0\}$ and $Q_2 = \{\omega_0, \omega_2, \omega_0\}$.

Because $\beta'$ is locally constant, and because the vertices of the chains $Q_1$ and $Q_2$ are of a special form, there exists an integer $N$ such that formula (2.2) becomes

$$h(\widetilde{Q}_1) = (\beta'(N,\omega^0) - \beta'(N,\omega^1)) + (\beta'(N,\sigma^{-N}\omega^1) - \beta'(N,\sigma^{-N}\omega^0)), \tag{2.4}$$

and

$$h(\widetilde{Q}_2) = (\beta(N,\omega^0) - \beta(N,\omega^2)) + (\beta(N,\sigma^{-N}\omega^2) - \beta(N,\sigma^{-N}\omega^1)). \tag{2.5}$$

Each term in formulas (2.4) and (2.5) can be computed using formula (2.1). For $j = 1, 2$, one has

$$h(\widetilde{Q}_j) = \sum_i c(C_i, \widetilde{Q}_j)\beta'(C_i),$$

where $C_i$'s are centered blocks defining the symmetric cylinders in the $N$-th Markov partition considered above, $c(C_i, \widetilde{Q}_j)$'s are integers, and $\beta'(C_i)$ is the value of $\beta'$ over $C_i$. Among the $C_i$'s we distinguish two blocks:

$$C_1 = sb_2sb_2sb_1s\ldots, \quad C_2 = sb_1sb_2sb_1s\ldots.$$

Note that the coefficients $c(C_1, \widetilde{Q}_1)$ and $c(C_2, \widetilde{Q}_2)$ are different from zero, while the coefficients $c(C_1, \widetilde{Q}_2)$ and $c(C_2, \widetilde{Q}_1)$ are zero.

Let $\alpha_1 = \beta(C_1)$ and $\alpha_2 = \beta(C_2)$. It is clear that by making arbitrarily small changes of the values $\alpha_1$ and $\alpha_2$, respectively, we can make the heights $h(\widetilde{Q}_1)$ and $h(\widetilde{Q}_2)$ different from zero. If both heights have the same sign, consider the quadrangle $-\widetilde{Q}_2$ instead of $\widetilde{Q}_2$. Note that this can be achieved by keeping the ratio $h_1/h_2$ irrational.

In the last part of the proof, we make $h(\widetilde{Q}_1)/h(\widetilde{Q}_2)$ irrational. We distinguish two cases.

CASE 1: A change in $\alpha_1$ and $\alpha_2$ keeps $h_1$ and $h_2$ fixed.

In this case, in order to make $h(\widetilde{Q}_1)/h(\widetilde{Q}_2)$ irrational, it is enough to make an arbitrarily small change of $\alpha_1$. After the perturbation the ratio $h_1/h_2$ is unchanged, so it is also irrational.

CASE 2: A change in $\alpha_1$ and $\alpha_2$ does not keep $h_1$ and $h_2$ fixed.

In this case, an arbitrarily small change of $\alpha_1$ or $\alpha_2$ makes the ratios $h(\widetilde{Q}_1)/h(\widetilde{Q}_2)$ and $h_1/h_2$ irrational. Indeed, without loss of generality we can assume that

$$h(\widetilde{Q}_1) = k_1 + l_1\alpha_1, \quad h(\widetilde{Q}_2) = k_2,$$
$$h_1 = \delta_1 + m_1\alpha_1, \quad h_2 = \delta_2 + m_2\alpha_1,$$
$$\frac{h_1}{h_2} \notin \mathbb{Q}, \quad \frac{h(\widetilde{Q}_1)}{h(\widetilde{Q}_2)} < 0, \quad \frac{h_1}{h_2} < 0,$$

where $k_1, k_2, \delta_1, \delta_2$ are non-zero real numbers determined by the values of $\beta$ over cylinders different from $C_1$, $l_1$ and $m_1$ are non-zero integers, and $m_2$ is an integer. Observe that for any $\epsilon > 0$ the set

$$S = \left\{ \alpha_1 \in (-\epsilon, \epsilon) \middle| \frac{k_1 + l_1\alpha_1}{k_2} \in \mathbb{Q}, \text{ or } \frac{\delta_1 + m_1\alpha_1}{\delta_2 + m_1\alpha_2} \in \mathbb{Q} \right\}$$

is countable with a possible exception if $\delta_1/\delta_2 = m_1/m_2$. But this is impossible, since for the initial perturbation made above one can assume that $h_1/h_2 \notin \mathbb{Q}$. So the set $(-\epsilon, \epsilon) - S$ contains elements $\alpha_1$ arbitrarily close to 0. One can now change $\beta'$ over $C_1$ and obtain the desired transformation. ∎

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